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# PHASE SPACE, WAVELET TRANSFORM AND TOEPLITZ–HANKEL TYPE OPERATORS*

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ABSTRACT

Let IG$(n)$ be the Euclidean group with dilations. It has a maximal compact subgroup SO$(n-1)$. The homogeneous space can be realized as the phase space IG$(n)$/SO$(n-1) \cong R^n \times R^n$. The square-integrable representation gives the admissible wavelets AW and wavelet transforms on $L^2(R^n)$. With Laguerre polynomials and surface spherical harmonics an orthogonal decomposition of AW is given; it turns to give a complete orthogonal decomposition of the $L^2$-space on the phase space $L^2(R^n \times R^n, dxdy/|y|^{n+1})$ of the form $\bigoplus_{k=0}^{\infty}\bigoplus_{l=0}^{\infty}\bigoplus_{j=0}^{a_l} A_{l,j}^k$. The Schatten–von Neumann properties of the Toeplitz–Hankel type operators between these decomposition components are established.

## 1. Introduction

The continuous wavelet transform in the one-dimensional case can be obtained in two ways: one from the theory of square-integrable group representation and the other from the Calderón representation formula.

Let $G$ be a locally compact group with left Haar measure $dx$. Let $x \to U(x)$ $(x \in G)$ be an irreducible unitary representation of $G$ in a Hilbert space $\mathcal{H}$. A vector $\psi \in \mathcal{H}$ is said to be admissible if it satisfies the following "admissibility condition":

$$(1.1) \qquad 0 < c_\psi := \int_G |(\psi, U(x)\psi)|^2 dx/(\psi,\psi) < \infty,$$

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where $(\cdot,\cdot)$ is the inner product of $\mathcal{H}$. We denote the set of all such vectors by AW. If AW$\neq \phi$, then the representation $U$ is called square-integrable. For $\psi \in$ AW, $f \to (f, U(x)\psi)$ is called "continuous wavelet transform".

A. Grossman and J. Morlet [GM] introduced the wavelet transform in the one-dimensional case, where the group $G$ is the affine group $ax+b$. Let

$$f_{(b,a)}(x) := U(b,a)f(x) = \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right)$$

be the representation of $G$ on the Hardy space $H^2(R)$. Then the (affine) wavelet transform $W_\psi$ for $f$ in $H^2(R)$ associated with an admissible wavelet $\psi$ is given by

$$W_\psi f(b,a) := (f, \psi_{(b.a)}) = \frac{1}{\sqrt{a}} \int_R \overline{\psi}\left(\frac{x-b}{a}\right) f(x)dx.$$

If $\psi \in AW$ and $\hat{\psi}(\xi) = 0$ for $\xi \leq 0$, then $\psi$ is called an admissible analyzing wavelet. For an admissible analyzing wavelet $\psi$, every $f \in H^2$ can be reconstructed from $W_\psi f(b,a)$:

$$f(x) = c_\psi^{-1} \int_G W_\psi f(b,a) \psi_{(b,a)}(x) \frac{dadb}{a^2}. \tag{1.2}$$

In fact, the above analysis by A. Grossman and J. Morlet was quite close to a technique developed by A. Calderón and his collaborators for the study of singular integral operators [C] in 1964. The basic tool is the so-called Calderón representation formula, that can be expressed as follows. Let $\psi(x)$ be a function such that $\psi \in L^1(R)$, $\hat{\psi}(-\xi) = \hat{\psi}(\xi)$ and

$$0 < c'_\psi := \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < \infty, \tag{1.3}$$

then for $f \in L^2(R)$, we have the Calderón formula:

$$f(x) = \frac{1}{2c'_\psi} \int_R (\psi_y * \tilde{\psi}_y * f(x)) \frac{dy}{y^2},$$

where

$$\psi_y(x) = \frac{1}{\sqrt{|y|}} \psi\left(\frac{x}{y}\right), \quad \tilde{\psi}(x) = \overline{\psi(-x)}.$$

If $\psi \in L^1(R)$, $\hat{\psi}(\xi) = 0$ for $\xi \leq 0$ and $\psi$ satisfies (1.3), then for $f \in H^2$ the Calderón formula becomes

$$f(x) = \frac{1}{c'_\psi} \int_R (\psi_y * \tilde{\psi}_y * f(x)) \frac{dy}{y^2}. \tag{1.4}$$

In fact, (1.3) is just the admissibility condition (1.1). For $f \in H^2$, $f \to \tilde{\psi}_y * f(x)$ is the wavelet transform and (1.2) is (1.4).

Let $U$ be the upper-half plane. In [JP1], [JP2], by an orthogonal decomposition of AW with Laguerre polynomials, orthogonal decompositions of $L^2(U, y^\alpha dxdy)$ were given in the cases $\alpha = -2$ and $\alpha > -1$ respectively. The Toeplitz–Hankel type operators between the decomposition components were defined, and boundedness, compactness and Schatten–von Neumann properties of them were established. In this paper, we want to consider the similar problems in the higher-dimensional case.

From the above discussion, we know that in the one-dimensional case the two different ways can induce the same results, i.e. (1.2) and (1.4). In the higher–dimensional case, since there is no concept of the "analyzing" in the definition of admissible wavelet, the above two ways will induce two different results. One is the Calderón representation formula, which induces a decomposition of $L^2(R^{n+1}, dxdy/|y|^{n+1})$, and the other is the wavelet transform associated with the square-integrable group representation, see [To]. Here we will introduce another kind of wavelet transform which induces a decomposition of $L^2(R^n \times R^n, dxdy/|y|^{n+1})$.

The $n$-dimensional generalization of the Calderón formula is quite simple. The wavelet $\psi$ is now a radial function in $L^1(R^n)$ such that

$$0 < c''_\psi := \mathrm{Vol}(S^{n-1}) \int_0^\infty |\hat{\psi}(a\xi)|^2 \frac{da}{a} = \int_{R^n} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|^n} < \infty$$

for all $\xi \neq 0$. Without confusing with the definition of $\psi_{(b,a)}(x)$ in the case $n = 1$, we also let

$$\psi_{(b,a)}(x) := a^{-n/2} \psi\left(\frac{x-b}{a}\right).$$

Then for $f \in L^2(R^n)$,

$$f(x) = \frac{1}{c''_\psi} \int_{R_+^* \times R^n} T_f(b,a) \psi_{(b,a)}(x) \frac{dadb}{a^{n+1}}, \tag{1.5}$$

where $R_+^* = (0, \infty)$ and $T_f(b,a)$ is the function of $n+1$ variables defined by:

$$T_f(b,a) := (f, \psi_{(b,a)}). \tag{1.6}$$

The map $f \to T_f(b,a)$ is the wavelet transform associated with $\psi$. In [JP3], we study this kind of wavelet transform associated with Hermite polynomials and we construct a series of wavelets. The ranges of this kind of wavelet transform of $L^2(R^n)$ with these wavelets form an orthogonal decomposition of $L^2(R^{n+1}, dxdy/|y|^{n+1})$.

The wavelet transform with the square-integrable group representation in the higher–dimensional case is associated with the group IG$(n)$, *the Euclidean group with dilations*, introduced by R. Murezin [Mu]. Namely,

$$\begin{aligned}\mathrm{IG}(n) :&= R^n \times R_+^* \times \mathrm{SO}(n)\\ &= \{g = (b, a, \rho) : b \in R^n, a > 0, \rho \in \mathrm{SO}(n)\},\end{aligned}$$

with the group law:

$$(b', a', \rho')(b, a, \rho) = (a'\rho' b + b', a'a, \rho'\rho).$$

The elements $g$ of IG$(n)$ can be written as $\begin{pmatrix} a\rho & b \\ 0 & 1 \end{pmatrix}$, and the above group operation becomes the product of matrices. IG$(n)$ has the left Haar measure $dg = a^{-n-1} da db d\rho$, where $d\rho$ is the Haar measure of SO$(n)$. Let $U_g$ be the irreducible unitary representation of IG$(n)$ on $L^2(R^n)$ defined by

$$U_g\psi(x) := \psi_g(x) = a^{-n/2}\psi(g^{-1}(x))$$

where

$$g = (b, a, \rho), \quad g(x) = a\rho(x) + b \quad \text{and} \quad g^{-1}(x) = \frac{1}{a}\rho^{-1}(x - b).$$

The admissibility condition is

$$(1.7) \qquad 0 < k'_\psi := \mathrm{Vol}(\mathrm{SO}(n-1)) \int_{R^n} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|^n} < \infty,$$

see [Mu], [To] or by a direct calculation. For $f \in L^2(R^n)$,

$$f(x) = \frac{1}{k'_\psi} \int_{R_+^* \times R^n \times SO(n)} T_f(b, a, \rho)\psi_{(b,a,\rho)}(x) \frac{da db d\rho}{a^{n+1}}$$

and the wavelet transform is now the function in $L^2(R_+^* \times R^n \times \mathrm{SO}(n))$ defined by

$$(1.8) \qquad T_f(b, a, \rho) := (f, \psi_{(b,a,\rho)}),$$

where $\psi_{(b,a,\rho)} := U_g\psi = a^{-n/2}\psi\left(\frac{1}{a}\rho^{-1}(x - b)\right)$.

The wavelet transforms (1.6) and (1.8) are functions of $n+1$ and $n(n+1)/2+1$ variables respectively, thus they cannot be considered as functions on the phase space. The functions on the phase space $R^n \times R^n$ of $R^n$ depend on $2n$ variables. We want now to generalize the affine wavelet transform to higher dimensions. In order that the generalized wavelet transform will be a mapping from $L^2(R^n)$ to the space of functions on the phase space $R^n \times R^n$, we will consider a wavelet transform associated with the square-integrable group representation modulo a subgroup (see [To]). We will consider here the quotient group $\mathrm{IG}(n)/\mathrm{SO}(n-1)$. Since

$$\begin{aligned}\mathrm{IG}(n)/\mathrm{SO}(n-1) &= R_+^* \times R^n \times \mathrm{SO}(n)/\mathrm{SO}(n-1) \\ &\cong R_+^* \times R^n \times S^{n-1} = R^n \times R^n,\end{aligned}$$

the wavelet transforms of $f \in L^2(R^n)$ are functions on $R^n \times R^n$. In this case, for $f \in L^2(R^n)$, it is also possible to reconstruct $f(x)$ from the corresponding wavelet transform. Let us give the definition of the wavelet transform.

Let $\omega = (1, 0, \ldots, 0)^t$ be a fixed point in $S^{n-1}$; here $\xi^t$ denotes the transpose of a vector $\xi$ in $R^n$. For any $\xi \in S^{n-1}$, there exists an element $\rho_\xi \in \mathrm{SO}(n)$ such that $\rho_\xi^{-1}\xi = \omega$. In fact there exists a family of such $\rho_\xi$, see [Vi], p. 437. Here for $\xi \in S^{n-1}$, only one element $\rho_\xi$ is corresponded in a fixed way and we define

$$\psi_{(b,a,\xi)}(x) := a^{-n/2}\psi\left(\frac{1}{a}\rho_\xi^{-1}(x-b)\right),$$

and the wavelet transform of $f \in L^2(R^n)$ to be

$$T_f(b,a,\xi) := (f, \psi_{(b,a,\xi)}). \tag{1.9}$$

Then for some functions $\psi$ (the admissible wavelets), the following reconstructing formula holds:

$$f(x) = \frac{1}{h_\psi}\int_{R_+^* \times R^n \times S^{n-1}} T_f(b,a,\xi)\psi_{(b,a,\xi)}(x)\frac{dadbd\sigma(\xi)}{a^{n+1}}, \tag{1.10}$$

where $d\sigma(\xi)$ is the normalized surface area measure on $S^{n-1}$. Taking Fourier transforms on both sides of (1.10) or by calculating

$$\int_{R_+^* \times R^n \times S^{n-1}} |(\psi, \psi_{(b,a,\xi)})|^2 \frac{dadbd\sigma(\xi)}{a^{n+1}},$$

we can get that the admissible condition (assuring (1.10) true) is

$$\int_0^\infty \int_{S^{n-1}} |\hat{\psi}(a\rho_\xi^{-1}b)|^2 \frac{dad\sigma(\xi)}{a} = h_\psi < \infty,$$

and $h_\psi$ is a constant independent of all $b \in S^{n-1}$. For radial functions $\psi$, this admissible condition is just (1.7). But for a general functon $\psi$, this condition is hard to verify. In order to give a good admissible condition as (1.7), we introduce another kind of wavelet transform. To do this, taking Fourier transforms of (1.9) with respect to the first variable $b$, we have

$$\hat{T}_f(\eta, a, \xi) = a^{n/2}\overline{\hat{\psi}(a|\eta|\rho_\xi^{-1}\eta')}\hat{f}(\eta), \tag{1.11}$$

with $\eta = |\eta|\eta'$.

For $(b, a, \xi) \in R^n \times R^*_+ \times S^{n-1}$, we define wavelet transform $W_\psi f(b, a, \xi)$ of $f \in L^2(R^n)$ via the Fourier transform with respect to the first variable,

$$(W_\psi)^\wedge f(\eta, a, \xi) := a^{n/2}\overline{\hat{\psi}(a|\eta|\rho_{\eta'}\xi)}\hat{f}(\eta), \tag{1.12}$$

where for $\eta \in R^n$, $\rho_{\eta'} \in SO(n)$ is given as above, i.e. $\eta = |\eta|\eta' = |\eta|\rho_{\eta'}\omega$. If $\psi$ is a radial function, then (1.12) is (1.9). Without confusion with the above, also denote $\psi_{(b,a,\xi)}$ by

$$\hat{\psi}_{(b,a,\xi)}(\eta) := a^{n/2}e^{-i\eta b}\hat{\psi}(a|\eta|\rho_{\eta'}\xi), \tag{1.13}$$

then $W_\psi f$ given by (1.12) also can be written as (1.9) with $\psi_{(b,a,\xi)}$ given by (1.13):

$$W_\psi f(b, a, \xi) = (f, \psi_{(b,a,\xi)}). \tag{1.14}$$

For this kind of wavelet transform, the admissible condition is the following:

$$\int_0^\infty \int_{S^{n-1}} |\hat{\psi}(a|\eta|\rho_{\eta'}\xi)|^2 \frac{da d\sigma(\xi)}{a} = c_\psi < \infty, \tag{1.15}$$

for all $\eta \in R^n/\{0\}$. Since $d\sigma(\xi)$ is invariant under the action of SO($n$), we have

$$\begin{aligned} \text{left side of (1.15)} &= \int_0^\infty \int_{S^{n-1}} |\hat{\psi}(a|\eta|\xi)|^2 \frac{da d\sigma(\xi)}{a} \\ &= \int_{R^n} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|^n}. \end{aligned}$$

Finally we know the admissible condition is still (1.7). Thus in the following, let

$$\mathrm{AW} = \left\{\psi : \psi \in L^2(R^n), 0 < k_\psi := \int_{R^n} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|^n} < \infty\right\},$$

and for $\psi \in$ AW, the wavelet transform of $f \in L^2(R^n)$ is given by (1.12).

In this paper, we give an orthogonal decomposition of AW to be $\overline{\text{span}}\{\psi^{k,l,j}\}$ with the help of Laguerre polynomials $L_k^{(\alpha)}(x)$ and the surface spherical harmonics $Y_j^l$. Then, using this decomposition and the wavelet transform given by (1.12), we construct an orthogonal decomposition of the $L^2$-space on the phase space $L^2(R^n \times R^n, dxdy/|y|^{n+1})$ of the form $\bigoplus_{k=0}^{\infty} \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{a_l} A_{l,j}^k$, where $A_{l,j}^k$ are the ranges of wavelet transform (1.12) of the orthogonal wavelets $\psi^{k,l,j}$ with $f \in L^2(R^n)$. We then study the boundedness, compactness and Schatten–von Neumann properties of the Topelitz–Hankel type operators between the components of this decomposition. We will construct the orthogonal decomposition and formulate main results about the operators in §2, and give the proofs in §3.

## 2. The space decomposition and the main results

First, let us construct an orthogonal decomposition of AW. Let $H_l$ be the space of all linear combinations of functions of the form $f(r)Y(x)$, where $f$ ranges over the radial functions and $Y$ ranges over the solid spherical harmonics of degree $l$. Then $L^2(R^n)$ can be decomposed as the orthogonal sum (see [SW], p. 151):

$$L^2(R^n) = \bigoplus_{l=0}^{\infty} H_l. \tag{2.1}$$

Every element $f_l$ in $H_l$ can be written in the form $\sum_{j=1}^{a_l} f_{l,j}(r)Y_j^l(x)$, and

$$\int_{R^n} |f_l(x)|^2 dx = \sum_{j=1}^{a_l} \int_0^{\infty} |f_{l,j}(r)|^2 r^{n-1} dr,$$

where

$$a_0 = 1, \quad a_1 = n, \quad a_l = \binom{n+l-1}{l} - \binom{n+l-3}{l-2}, \quad l \geq 2$$

and $\{Y_j^l(x)\}_{j=1}^{a_l}$ is an orthogonal basis of the space $\mathcal{H}_l$ of surface spherical harmonics of degree $l$ (see [SW] p. 140). It is well known for $n = 2$, $a_k = 2$ and

$$Y_1^k(x) = \frac{\cos k\theta}{\sqrt{\pi}}, \quad Y_2^k(x) = \frac{\sin k\theta}{\sqrt{\pi}} \quad \text{with } x = e^{i\theta}.$$

For $n \geq 3$, let us give the orthogonal basis $Y_j^l$ of $\mathcal{H}_l$. For $x \in S^{n-1}$, $x = (x_1, \ldots, x_n)$, let $\gamma_{n-j}^2 = x_1^2 + \cdots + x_{n-j}^2$. Write $x$ in spherical coordinates:

$$\frac{x_{n-j}}{\gamma_{n-j}} = \cos\theta_{n-j-1}, \quad \frac{\gamma_{n-j-1}}{\gamma_{n-j}} = \sin\theta_{n-j-1}$$

with $0 \le \theta_1 \le 2\pi, 0 \le \theta_\nu \le \pi$ for $2 \le \nu \le n-1$, then

$$d\sigma(x) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \sin^{n-2}\theta_{n-1}\cdots\sin\theta_2 d\theta_1\cdots d\theta_{n-1}.$$

Let $C_k^p(t)$ be the Gegenbauer polynomials of degree $k$; they can be written as

$$C_k^p(t) = \frac{2^k\Gamma(p+k)}{k!\Gamma(p)}\left[t^k - \frac{k(k-2)}{2^2(p+k-1)}t^{k-2} + \frac{k(k-1)(k-2)(k-3)}{2^4 1\cdot 2\cdots(p+k-1)(p+k-2)}t^{k-4} + \cdots\right].$$

It is known that

$$\left\{2^{p-1}\Gamma(p)\left[\frac{2(k+p)\cdot k!}{\Gamma(2p+1)\pi}\right]^{\frac{1}{2}} C_k^p(t)\right\}_{k=0}^{\infty}$$

is an orthonormal basis on the segment $[-1,1]$ relative to the weight $(1-t^2)^{p-\frac{1}{2}}$. Let

$$G_K^l(x) := A_K^l \Pi_{s=0}^{n-3} C_{k_s-k_{s+1}}^{\frac{n-s-2}{2}+k_{s+1}}(\cos\theta_{n-s-1})\sin^{k_{s+1}}\theta_{n-s-1}e^{\pm ik_{n-2}\theta_1}, \tag{2.2}$$

where $K = (k_1, \ldots, \pm k_{n-2})$, $l = k_0 \ge k_1 \ge \cdots \ge k_{n-2} \ge 0$ and $A_K^l$ is a normalization constant. Then for $n \ge 3$, the canonical basis $Y_j^l(x)$ in $\mathcal{H}_l$ is the rearrangement of $G_K^l$ in the following order: $G_K^l(x)$, $K = (k_1, \ldots, \pm k_{n-2})$ precedes $G_M^l(x)$, $M = (m_1, \ldots, \pm m_{n-2})$ if there is an $s$ such that $k_i = m_i$, $0 \le i \le s$ and $k_{s+1} < m_{s+1}$ (or $\pm k_{n-2} < \pm m_{n-2}$). For $Y_k^l$ and $G_K^l$, see [Vi] pp. 457–468.

If $\psi \in \mathrm{AW} \subset L^2(R^n)$, then we can write

$$\hat{\psi}(\xi) = \sum_{l=0}^{\infty}\sum_{j=1}^{a_l} c_l f_{l,j}(r) Y_j^l(\xi),$$

$$\frac{\hat{\psi}(\xi)}{|\xi|^{n/2}} = \sum_{l=0}^{\infty}\sum_{j=1}^{a_l} \frac{c_l f_{l,j}(r)}{r^{n/2}} Y_j^l(\xi),$$

and

$$\int_{R^n} \frac{|\hat{\psi}(\xi)|^2}{|\xi|^n} d\xi = \sum_{l=0}^{\infty}\sum_{j=1}^{a_k} c_l^2 \int_0^{\infty} |f_{l,j}(r)|^2 \frac{dr}{r}. \tag{2.3}$$

For $\alpha > -1$, let $L_k^{(\alpha)}(x) = \sum_{\nu=0}^{k}\binom{k+\alpha}{k-\nu}(-x)^\nu/\nu!$ be the Laguerre polynomials (see [Sz]). They satisfy

$$\int_0^{\infty} e^{-x}x^{\alpha}L_k^{(\alpha)}(x)L_{k'}^{(\alpha)}(x)dx = \Gamma(\alpha+1)\binom{k+\alpha}{k}\delta_{kk'}. \tag{2.4}$$

For $k, l \in Z^+, 1 \leq j \leq a_l$, let $\psi^{k,l,j}$ be the functions on $R^n$, defined via their Fourier transforms:

$$\hat{\psi}^{k,l,j}(\xi) := |2\xi|^{\frac{\alpha+1}{2}} e^{-|\xi|} L_k^{(\alpha)}(2|\xi|) Y_j^l(\xi). \tag{2.5}$$

By (2.3), Lemma 2.18 in [SW] and by Theorem 5.7.1 in [Sz],

$$\{e^{-x} x^{\frac{\alpha}{2}} L_k^{(\alpha)}(x)\}_{k=0}^{\infty}$$

is complete in $L^2(0,\infty)$. Thus we have

$$\mathrm{AW} = \overline{\mathrm{span}}\{\psi^{k,l,j}\}_{\{k\geq 0, l\geq 0, 1\leq j\leq a_l\}}.$$

Let us denote $L^2 := L^2(R^n \times R^n, dxdy/|y|^{n+1})$. For $\psi \in \mathrm{AW}$, let $W_\psi$ be the operator (wavelet transform) from $L^2(R^n)$ into $L^2$ defined by (1.12). Let $A_\psi$ denote the range of $W_\psi$, i.e.

$$\begin{aligned} A_\psi := \{W_\psi f(x,y) = W_\psi f(x,|y|,y') :\ & x \in R^n, \\ & \text{and } y = |y|y' \in R^n, y' \in S^{n-1}, f \in L^2(R^n)\}. \end{aligned} \tag{2.6}$$

Let $\tau$ be the operator from $A_\psi$ onto $L^2(R^n)$ defined by

$$\tau(F)(x) := k_\psi^{-1}(2\pi)^{-n} \int_{R^n \times R_+^* \times S^{n-1}} \hat{F}(\eta,a,\xi) e^{i\eta x} \hat{\psi}(a|\eta|\rho_{\eta'}\xi) \frac{da d\eta d\sigma(\xi)}{a^{n+1}}, \tag{2.7}$$

where $\hat{F}(\eta,a,\xi)$ denotes the Fourier transform of $F(b,a,\xi)$ to the first variable. Then for $\psi \in \mathrm{AW}$, $\tau W_\psi = I$ on $L^2(R^n)$, this is just the reconstructing formula.

The space $A_\psi$ has a reproducing kernel, denoted by $K(g,g_1) = K_{g_1}(g)$ with $g_1 = (x_1,a_1,\xi_1), g = (x,a,\xi)$. Let $W_\psi f$ be given by (1.14). By the reconstructing formula

$$f(x) = k_\psi^{-1} \int_{R_+^* \times R^n \times S^{n-1}} (f, \psi_{(b,a,\xi)}) \psi_{(b,a,\xi)}(x) \frac{dadbd\sigma(\xi)}{a^{n+1}},$$

we have

(2.8)

$$(f, \psi_{(b_1,a_1,\xi_1)}) = k_\psi^{-1} \int_{R_+^* \times R^n \times S^{n-1}} (f, \psi_{(b,a,\xi)})(\psi_{(b,a,\xi)}, \psi_{(b_1,a_1,\xi_1)}) \frac{dadbd\sigma(\xi)}{a^{n+1}}.$$

Thus by (2.8),

$$K_{g_1}(g) = k_\psi^{-1}(\psi_{(b_1,a_1,\xi_1)}, \psi_{(b,a,\xi)}).$$

Taking the Fourier transform with respect to the variable $b$, we have

$$\hat{K}_{g_1}(\eta,a,\xi) = \frac{1}{k_\psi}\overline{\hat{\psi}(a|\eta|\rho_{\eta'}\xi)}\hat{\psi}_{(b_1,a_1,\xi_1)}(\eta)(aa_1)^{n/2} = \frac{1}{k_\psi}\overline{\hat{\psi}(a|\eta|\rho_{\eta'}\xi)}\hat{\psi}(a_1|\eta|\rho_{\eta'}\xi_1)e^{-i\eta b_1}(aa_1)^{n/2}. \tag{2.9}$$

If $\psi = \psi^{k,l,j}$ is defined by (2.5), we obtain operators $T^{k,l,j} := W_{\psi^{k,l,j}}, \tau^{k,l,j}$ and subspaces $A^k_{l,j}$ from (1.12), (2.7) and (2.6) respectively.

From the orthogonality of $L_k^{(\alpha)}(x)$ and $Y_j^l(\xi)$, we know $A^k_{l,j}$ are mutually orthogonal subspaces of $L^2$. Moreover, we have

THEOREM 1: *Let $A^k_{l,j}$ be the subspaces defined by* (2.6) *with $\psi = \psi^{k,l,j}$, then*

$$L^2(R^n \times R^n, dxdy/|y|^{n+1}) = \bigoplus_{k=0}^{\infty}\bigoplus_{l=0}^{\infty}\bigoplus_{j=1}^{a_l} A^k_{l,j}.$$

From (2.9), we know that each $A^k_{l,j}$ has reproducing kernel:

$$\hat{K}^{k,l,j}_{g_1}(\eta,a,\xi) = \frac{(a_1a)^{n/2}}{k_{\psi^{k,l,j}}}\overline{\hat{\psi}^{k,l,j}(a|\eta|\rho_{\eta'}\xi)}\hat{\psi}^{k,l,j}(a_1|\eta|\rho_{\eta'}\xi_1)e^{-i\eta b_1}. \tag{2.10}$$

We can calculate

$$k^2_{\psi^{k,l,j}} = \frac{\Gamma(k+\alpha+1)}{k!}.$$

Let $P^k_{l,j}$ be the orthogonal projection from $L^2$ onto $A^k_{l,j}$; then for $(b,a,\xi) \in R^n \times R^*_+ \times S^{n-1}$, we have

$$(P^k_{l,j}F)(b,a,\xi) = \int_{R^n\times R^*_+\times S^{n-1}} K^{k,l,j}_{g_1}(\eta,a,\xi)F(g_1)dg_1, \tag{2.11}$$

where

$$dg_1 = \frac{db_1da_1d\sigma(\xi_1)}{a_1^{n+1}} \quad \text{with } g_1 = (b_1,a_1,\xi_1) \quad \text{for all } F \in L^2.$$

We now define the Toeplitz–Hankel type operators:

$$T^{k,l,j}_{b,k',l',j'} := P^k_{l,j}M_bP^{k'}_{l',j'}, \tag{2.12}$$

where $M_b$ is the multiplication operator by $b$ and $b(x,a) = b(x,a,0)$ is a function on $R^n \times R^*_+ \times S^{n-1}$ restricted to $R^n \times R^*_+$, defined by its Fourier transform with respect to the first variable $x$:

$$\hat{b}(\cdot,a)(\xi) := e^{-|\xi|a}\hat{b}(\xi). \tag{2.13}$$

Let $B_p$ be the Besov space $B_p^{\frac{n}{p},p}(R^n)$. In what follows $b \in B_p$ means its boundary value $b(x)$, defined by (2.13), is in $B_p$. Let $S_p$ be the Schatten-von Neumann class. See [JP], [P] for information concerning the space $B_p$ and $S_p$. We have

THEOREM 2: *Let $T^{k,l,j}_{b,k',l',j'}$ be the operator defined by (2.12), then:*
(1) *If $l \neq l'$, then $T^{k,l,j}_{b,k',l',j'} \equiv 0$.*
(2) *If $l = l'$ and $k = k'$, then $T^{k,l,j}_{b,k',l',j'} \in S_\infty$ iff $b \in L^\infty$; $T^{k,l,j}_{b,k',l',j'}$ is never compact unless it is zero.*
(3) *If $l = l', k \neq k'$ and $\frac{n}{|k-k'|} < p \leq \infty$, then $T^{k,l,j}_{b,k',l',j'} \in S_p$ iff $b \in B_p$.*
(4) *If $l = l', k \neq k'$ and $0 < p \leq \frac{n}{|k-k'|}$ and $T^{k,l,j}_{b,k',l',j'} \in S_p$, then $b = 0$.*

*Remark:* From Theorem 2, we know that the cut-off phenomenon of $T^{k,l,j}_{b,k',l',j'}$ depends only on $k$ and $k'$, and it happens at the point $\frac{n}{|k-k'|}$.

## 3. The proof of Theorem 2

Applying (2.11) and (2.12) to $F(x,y,\rho) = W_{\psi^{k',j',l'}} f(x,y,\rho) \in A^{k'}_{l',j'}$, we have

$$(T^{k,l,j}_{b,k',l',j'}F)(x,y,\rho) = \int_{R^n\times R^*_+\times S^{n-1}} K^{k,l,j}_{g_1}(g)[b(g_1)W_{\psi^{k',j',l'}} f(g_1)]dg_1,$$

with $g = (x,y,\rho)$ and $g_1 = (x_1,y_1,\rho_1)$. Taking the Fourier transform with respect to the first variable $x$, we have

$$\begin{aligned}
&(T^{k,l,j}_{b,k',l',j'}F)^\wedge(\xi,y,\rho)\\
&= \frac{y^{\frac{n}{2}}}{k_{\psi^{k,l,j}}} \int_{R^n\times R^*_+\times S^{n-1}} \overline{\hat{\psi}^{k,l,j}(y|\xi|\rho_{\xi'}\rho)}\hat{\psi}^{k,l,j}(v|\xi|\rho_{\xi'}\rho_1)e^{-i\xi x_1}\cdot\\
&\quad [b(x_1,v)W_{\psi^{k',l',j'}} f(x_1,v,\rho_1)]\frac{v^{\frac{n}{2}}dx_1dvd\sigma(\rho_1)}{v^{n+1}}\\
&= \frac{y^{\frac{n}{2}}\overline{\hat{\psi}^{k,l,j}(y|\xi|\rho_{\xi'}\rho)}}{k_{\psi^{k,l,j}}} \int_{R^n\times R^*_+\times S^{n-1}} \hat{\psi}^{k,l,j}(v|\xi|\rho_{\xi'}\rho_1)\cdot\\
&\quad \frac{1}{(2\pi)^n}\int \hat{b}(\cdot,v)(\xi-\eta)f(\eta)\overline{\psi^{k',l',j'}(v|\eta|\rho_{\eta'}\rho_1)}d\eta\frac{dvd\sigma(\rho_1)}{v}\\
&= \frac{y^{\frac{n}{2}}\overline{\hat{\psi}^{k,l,j}(y|\xi|\rho_{\xi'}\rho)}}{(2\pi)^n k_{\psi^{k,l,j}}} \int_{R^n} \hat{f}(\eta)\hat{b}(\xi-\eta)A^{k,l,j}_{k',l',j'}(\xi,\eta)d\eta,
\end{aligned}$$

where $A^{k,l,j}_{k',l',j'}(\xi,\eta)$ is given by

$$(3.1)\qquad \begin{aligned} &A^{k,l,j}_{k',l',j'}(\xi,\eta)\\ =&\int_{S^{n-1}}\int_0^\infty \hat{\psi}^{k,l,j}(v|\xi|\rho_{\xi'}\rho_1)e^{-v|\xi-\eta|}\overline{\psi^{k',l',j'}(v|\eta|\rho_{\eta'}\rho_1)}\frac{dvd\sigma(\rho_1)}{v\cdot}\\ =&\int_0^\infty e^{-v(|\xi|+|\eta|+|\xi-\eta|)}L_k^{(\alpha)}(2v|\xi|)L_{k'}^{(\alpha)}(2v|\eta|)(4|\xi||\eta|)^{\frac{\alpha+1}{2}}v^\alpha dv\cdot\\ &\int_{S^{n-1}}Y_j^l(\rho_{\xi'}\rho_1)\overline{Y_{j'}^{l'}(\rho_{\eta'}\rho_1)}d\sigma(\rho_1)\\ =&\frac{(4|\xi||\eta|)^{(\alpha+1)/2}}{(|\xi|+|\eta|+|\xi-\eta|)^{\alpha+1}}A^{k,k'}(\xi,\eta)\cdot B^{(l,j)(l',j')}(\xi',\eta'). \end{aligned}$$

Here $\xi=|\xi|\xi',\eta=|\eta|\eta'$ and

$$(3.2)\qquad \begin{aligned} &A^{k,k'}(\xi,\eta)=\\ &\int_0^\infty L_k^{(\alpha)}(\frac{2|\xi|}{|\xi|+|\eta|+|\xi-\eta|}v)L_{k'}^{(\alpha)}(\frac{2|\eta|}{|\xi|+|\eta|+|\xi-\eta|}v)e^{-v}v^\alpha dv, \end{aligned}$$

and

$$(3.3)\qquad B^{(l,j)(l',j')}(\xi',\eta')=\int_{S^{n-1}}Y_j^l(\rho_{\xi'}\rho_1)\overline{Y_{j'}^{l'}(\rho_{\eta'}\rho_1)}d\sigma(\rho_1).$$

Thus we know that $T^{k,l,j}_{k',l',j'}$ is a vector-valued paracommutator (see [AFP]), and we can transform it into an ordinary paracommutator. Let $\tau^{k,l,j}$ and $T^{k',l',j'}$ be the operators from $A^k_{l,j}$ onto $L^2(R^n)$ and from $L^2(R^n)$ onto $A^{k'}_{l',j'}$ respectively defined in section 2. Let $t^{k,l,j}_{b,k',l',j'}$ be the operator from $L^2(R^n)$ onto itself defined by

$$(t^{k,l,j}_{b,k',l',j'}f)(x):=(\tau^{j,k,l}T^{k,l,j}_{b,k',l',j'}T^{k',l',j'}f)(x)$$

where $f\in L^2(R^n)$. Then we can get (omitting the details)

$$(3.4)\qquad (t^{k,l,j}_{b,k',l',j'}f)^\wedge(\xi)=k^{-1}_{\psi^{k,l,j}}\frac{1}{(2\pi)^n}\int_{R^n}\hat{f}(\eta)\hat{b}(\xi,\eta)A^{k,l,j}_{k',l',j'}(\xi,\eta)d\eta.$$

Thus $t^{k,l,j}_{b,k',l',j'}$ is a paracommutator with kernel $k^{-1}_{\psi^{k,l,j}}A^{k,l,j}_{k',l',j'}(\xi,\eta)$. Now let us estimate the degree of the vanishing of $A^{k,l}(\xi,\eta)$ as $\eta\to\xi$. Using the definition of $L_k^{(\alpha)}(x)$, we get

$$\begin{aligned}
(3.5)\qquad &\int_0^\infty y^\alpha e^{-y} L_k^{(\alpha)}(ay) L_l^{(\alpha)}(by)dy \\
=&\Gamma(k+\alpha+1)\Gamma(l+\alpha+1)\cdot \\
&\sum_{j=1}^{min}(k,l)\frac{b^j(1-b)^{l-j}}{(l-j)!j!}\frac{a^j(1-a)^{k-j}}{(k-j)!\Gamma(\alpha+j+1)} \\
=&\frac{\Gamma(k+\alpha+1)\Gamma(l+\alpha+1)}{\Gamma(\alpha+1)l!k!}(1-a)^k(1-b)^l\cdot \\
&{}_2F_1\left(-l,-k;\alpha+1;\frac{ab}{(1-a)(1-b)}\right) \\
=&\frac{\Gamma(\alpha+1+k+l)}{\Gamma(l+\alpha)l!k!}(1-a)^k(1-b)^l\cdot \\
&{}_2F_1\left(-l,-k;-k-l-\alpha;\frac{1-a-b}{(1-a)(1-b)}\right).
\end{aligned}$$

Let

$$a := \frac{2|\xi|}{|\xi|+|\eta|+|\xi-\eta|} \quad \text{and} \quad b := \frac{2|\eta|}{|\xi|+|\eta|+|\xi-\eta|}.$$

Then

$$(3.6)\qquad 1-a = \frac{|\eta|-|\xi|+|\xi-\eta|}{|\xi|+|\eta|+|\xi-\eta|} \sim \frac{|\xi-\eta|}{|\xi|+|\eta|},$$

$$(3.7)\qquad 1-b = \frac{|\xi|-|\eta|+|\xi-\eta|}{|\xi|+|\eta|+|\xi-\eta|} \sim \frac{|\xi-\eta|}{|\xi|+|\eta|}.$$

If $k \geq k'$, then by (3.5),

$$\begin{aligned}
&A^{k,k'}(\xi,\eta) = \frac{\Gamma(\alpha+k'+k+l)}{\Gamma(l+\alpha)k'!k!}(1-a)^{k-k'}\cdot \\
(3.8)\qquad &\sum_{\nu=0}^{k'}\frac{(-k')_\nu(-k)_\nu}{(-k-k'-\alpha)_\nu \nu!}(1-a-b)^\nu(1-a)^{k'-\nu}(1-b)^{k'-\nu} \\
&= (1-a)^{k-k'}\left\{\frac{(-1)^{k'}\Gamma(\alpha+k+1)}{\Gamma(l+\alpha)(k-k')!k'!} + c_1(1-a)(1-b)P_1(a,b)+\cdots\right\}.
\end{aligned}$$

If $k < k'$, then also by (3.5)

$$\begin{aligned}
(3.9)\qquad &A^{k,k'}(\xi,\eta) = \\
&(1-b)^{k'-k}\left\{\frac{(-1)^k\Gamma(\alpha+k'+1)}{\Gamma(k+\alpha)(k'-k)!k!} + c_2(1-a)(1-b)P_2(a,b)+\cdots\right\}.
\end{aligned}$$

In (3.8) and (3.9), $P_1(a,b)$ and $P_2(a,b)$ are polynomials in $a, b$.

Concerning $B^{(l,j)(l',j')}(\xi',\eta')$, from (3.3) and the fact that $d\sigma(\rho_1)$ is invariant under the action of SO$(n)$, we have

$$B^{lj,l'j'}(\xi',\eta') = \int_{S^{n-1}} Y_j^l(\rho_{\xi'}\rho_{\eta'}^{-1}\rho)\overline{Y_{j'}^{l'}(\rho)}d\sigma(\rho). \tag{3.10}$$

Since $Y_j^l(\rho_{\xi'}\rho_{\eta'}^{-1}\rho) \in \mathcal{H}_l$, then by the orthogonality of $Y_j^l, Y_{j'}^{l'}$, we have $B^{(j,l)(j',l')}(\xi',\eta') = 0$ for $l \neq l'$, thus (1) of Theorem 2 is true.

If $l = l'$, denote $\gamma = \rho_{\eta'}\rho_{\xi'}^{-1} \in$ SO$(n)$. By the relation of $Y_j^l, G_K^l$ given in section 2, we assume $Y_j^l = G_K^l, Y_{j'}^l = G_M^l$ with

$$K = (k_1, \ldots, k_{n-3}, \pm k_{n-2}), \quad M = (m_1, \ldots, m_{n-3}, \pm m_{k-2}).$$

Then

$$G_K^l(\gamma^{-1}\rho) = \sum_N t_{KN}^{nl}(\gamma)G_N^l(\rho),$$

see [Vi], p. 469. Thus

$$B^{(j,l)(j',l)}(\xi',\eta') = \int_{S^{n-1}} G_K^l(\gamma^{-1}\rho)\overline{G_M^l(\rho)}d\sigma(\rho) = t_{KM}^{nl}(\gamma),$$

with $\gamma = \rho_{\eta'}\rho_{\xi'}^{-1}$. Except for a finite number of points $\xi', \eta'$ in $S^{n-1}$, $t_{KM}^{nl}(\gamma) \neq 0$, as the case $K = 0, t_{0M}^{nl}(\gamma)$ can be expressed by Gegenbauer polynomials and the cosine of the Euler angles of $\gamma$. For $k \neq k'$, by these discussions and (3.6), (3.7), (3.8), (3.9), we know that $t_{b,k',l',j'}^{k,l,j}$ satisfy $A1, A2, A3(|k-k'|)$ and $A4\frac{1}{2}$ in [JP], [P], thus (3) and (4) of Theorem 2 is true by the theory of paracommutators (cf. [JP], [P]). If $k = k'$, by (3.8) or (3.9) and similar discussions, we can get (2) of Theorem 2.

*Remark:* In the definition of the Toeplitz–Hankel type operator $T_{b,k',l',j'}^{k,l,j}$, the symbol $b$ is defined by (2.13). In fact, we can consider a more general symbol, e.g. $b(x,a,\rho)$ is given by

$$\hat{b}(\xi,a,\rho) = e^{-|\xi|a}Y_{j_1}^{l_1}(\rho),$$

or

$$\hat{b}(\xi,a,\rho) = L_{k_1}^{(\alpha)}(2|\xi|a)e^{-|\xi|a}Y_{j_1}^{l_1}(\rho),$$

and we can establish similar results to Theorem 2.

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